Exam Advanced Logic

April 3rd, 2018

Instructions:

- Put your student number on the first page and subsequent pages (not your name, in order to allow for anonymous grading).
- Do not use pencil (or an erasable pen) or a red pen to make your exam.
- Motivate all your answers.
- Please leave open 10 lines on the first page to give us room for writing down the points.
- Your exam grade is computed as min(10, (the sum of all your points + 10) divided by 10). For example, someone who would only fill in a student number would get a 1, while someone who would make questions 1-9 perfectly but would skip the bonus question would get min(10, (90+10)/10) = 10.

Good luck!

1. **Induction (10 pt)** We define a new binary connective $\uparrow$. The truth definition of $\uparrow$ is given by: for all valuations v, $v(A \uparrow B) = 0$ iff $v(A) = v(B) = 1$ and $v(A \uparrow B) = 1$ otherwise. So for all well-formed formulas A, B of the language of propositional logic, $A{\uparrow}B$ is logically equivalent to $\neg(A \wedge B)$.

   Now let $\mathscr{L}_{\uparrow}$ be an alternative language of propositional logic based on the operator $\uparrow$ only (so without $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$).

   (a) Give an inductive definition of the well-formed formulas of $\mathscr{L}_{\uparrow}$.

   (b) Show by induction that $\{\uparrow\}$ is functionally complete, i.e. that every well-formed formula P of propositional logic is logically equivalent to a formula P' in $\mathscr{L}_{\uparrow}$. Prove this in two steps:

      i. Define P' by induction on well-formed formulas of propositional logic.

      ii. Prove by induction that for every formula P of propositional logic, P' is logically equivalent to P. Here, you may assume that it has already been proved that for every well-formed propositional formula P, the formula P' is in $\mathscr{L}_{\uparrow}$.

2. **Three-valued logics (10 pt)** Using a truth table, determine whether the following inference holds in $\mathbf{RM}_3$:

$$\models_{RM_3} (p \supset q) \vee (q \supset p)$$

   Write out the full truth table and do not forget to draw a conclusion.

3. **Tableaux for FDE and related many-valued logics (10 pt)** By constructing a suitable tableau, determine whether the following inference is valid in $\mathbf{K}_3$. If the inference is invalid, provide a counter-model.

$$\neg q \vee p \vdash_{K_3} (\neg p \vee q) \vee (p \wedge \neg q)$$

   NB: Do not forget to draw a conclusion from the tableau.

4. **Fuzzy logic (10 pt)** Determine whether the following holds in the fuzzy logic with set of designated values $D_{0.6} = \{x : x \geq 0.6\}$. If so, explain why. If not, provide a counter-model and show why it is one.

$$(p \wedge q) \rightarrow r \models_{0.6} (p \rightarrow r) \rightarrow (q \rightarrow r)$$

5. **Basic modal tableau (10 pt)** By constructing a suitable tableau, determine whether the following is valid in K. If the inference is invalid, provide a counter-model.

$$\Diamond\Box\Diamond p \vdash_K \Diamond\Box p.$$

NB: Do not forget to draw a conclusion from the tableau.

6. **Normal modal tableau (10 pt)** By constructing a suitable tableau, determine whether the following tense-logical inference is valid in $K^t_{\delta\tau}$ (dense, transitive). If the inference is invalid, provide a counter-model.

$$\langle F\rangle[P]q \ \vdash_{K^t_{\delta\tau}} \langle F\rangle q \wedge [P][P]q$$

NB: Do not forget to draw a conclusion from the tableau.

7. **Soundness and completeness (10pt)** Consider the following tableau in K_σ, which contains only one branch which we call b:

$$\begin{array}{c} \neg(\Diamond\Diamond p \supset \Diamond p), 0 \\ \Diamond\Diamond p, 0 \\ \neg\Diamond p, 0 \\ \Box\neg p, 0 \\ 0r1 \\ \Diamond p, 1 \\ 1r2 \\ p, 2 \\ \neg p, 1 \\ 1r0 \\ 2r1 \end{array}$$

   (a) Is b a *complete* branch? Briefly explain your answer.

   (b) Branch b is *open*. Provide an interpretation $\mathcal{I}$ and a function f such that f shows that $\mathcal{I}$ is *faithful* to b. Explain your answer.

8. **First-order modal tableau, variable domain (10 pt)** By constructing a suitable tableau, determine whether the following is valid in VK. If the inference is invalid, provide a counter-model.

$$\Box\forall x Px \wedge \Diamond\exists x Qx \vdash_{VK} \Diamond\exists x(Px \wedge Qx)$$

NB: Do not forget to draw a conclusion from the tableau.

9. **Default logic (10 pt)** The following translation key is given:

$L(x)$	x likes linear algebra
$T(x)$	x likes tableaux
$M(x)$	x studies mathematics
$A(x)$	x studies AI
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Consider the following set of default rules:

$$D = \left\{ \delta_1 = \frac{L(x) : M(x)}{M(x)}, \qquad \delta_2 = \frac{L(x) \wedge A(x) : T(x)}{T(x)}, \qquad \delta_3 = \frac{M(x) \vee T(x) : \neg A(x) \vee \neg M(x)}{\neg A(x)} \right\},$$

and initial set of facts:

$$W = \{A(d), L(d)\}.$$

This exercise is about the default theory $T = (W, D)$; so you only need to apply the defaults to the relevant constant d.

(a) Of each of the following sequences, state whether it is a *process*; and if so, whether or not the process is *closed*, and whether or not it is *successful*. Briefly explain your answers.

i. (δ_2)

ii. (δ_2, δ_1)

iii. (δ_2, δ_3)

(b) Draw the process tree of the default theory (W, D).

(c) What are the extensions of (W, D)?

10. **Bonus (10 pt)**

Consider the language of FDE, containing the connectives $\neg, \vee, \wedge$ (and not containing $\supset$). Does the following statement hold for all wffs A, B in that language?

"If $A \vdash_{K_3} B$ and $A \vdash_{LP} B$, then $A \vdash_{FDE} B$"

If yes, please explain exactly why the statement holds for all wffs A, B in the language of FDE.
If no, please provide a pair of wffs A, B and show in detail why that pair forms a counterexample to the statement.